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LETTER TO THE EDITOR

Bond percolation and the Yang–Lee edge singularity problems in three dimensions

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Abstract. Field theories with a trilinear interaction are used to describe both the random bond percolation problem and the Yang–Lee edge singularity problems in three dimensions at criticality. Renormalisation group functions are calculated to order four for the exponents η, ω and the beta function, and to order five for the remaining critical exponents. Estimations of the critical coupling constant and the exponents for percolation theory are found by summing conformal transformations of the Borel transformed series.

All the quantities of interest in the random bond problem on a lattice are attainable from the Potts model in which the number of states S per site is 1. To calculate, for instance, the probability that two bonds belong to the same cluster, one calculates the spin–spin correlation function for the Potts model with general S and then specialises with $S=1$. Essam (1980) has reviewed the random percolation problems and their connection with the Potts model and the interested reader is referred to that article. It is now well established (Zia and Wallace 1975, Priest and Lubensky 1976, Amit 1976) that the Potts model may be written in terms of a Euclidean field theory with a trilinear interaction, at least near the critical region. Certain global properties of the ϕ^3 theory have been investigated in the renormalisation group context by Zia and Wallace and recently the critical exponents of the theory were calculated to order three in the ε expansion, where $\varepsilon = d_c - d$ and the critical dimension $d_c = 6$ (de Alcantara Bonfim *et al* 1980, 1981). Also Houghton *et al* (1978) confirmed the validity of the ϕ^3 field theory in the percolation limit and analysed the high-order behaviour of the correlation functions near the critical dimension for the massless case.

The particular model we consider here has the Hamiltonian

$$\mathscr{H} = \tfrac{1}{2}\int \mathrm{d}^d q\, (q^2 + m_0^2)\phi_i(q)\phi_i(-q) + \frac{1}{3!}\lambda d_{ijk}\int \mathrm{d}^d q_1\, \mathrm{d}^d q_2\, \phi_i(q_1)\phi_j(q_2)\phi_k(-q_1-q_2). \qquad (1)$$

For the Potts problem d_{ijk} is a tensor represented by

$$d_{ijk} = \sum_{\alpha=1}^{n+1} e_i^\alpha e_j^\alpha e_k^\alpha \qquad (2)$$

where the $n+1$ vectors e_i^α $(\alpha = 1, 2, \ldots, n+1)$ of n components $(i = 1, 2, \ldots, n)$ satisfy

$$\sum_{\alpha=1}^{n+1} e_i^\alpha = 0, \qquad (3)$$

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$$\sum_{\alpha=1}^{n+1} e_i^\alpha e_j^\alpha = (n+1)\delta_{ij}, \tag{4}$$

$$\sum_{i=1}^{n} e_i^\alpha e_i^\beta = (n+1)\delta^{\alpha\beta} - 1. \tag{5}$$

All the correlation functions of interest are first calculated for general n and only then do we put $n = 0$ to obtain their values for the percolation problem unambiguously.

Another situation described by the same field theory is the Yang–Lee problem in which the density $\mathscr{D}(H)$ of zeros of the partition function of the Ising model are located on the imaginary magnetic field axis and

$$\mathscr{D}(H) \sim |H - H_0|^\sigma \tag{6}$$

as $H \to$ the critical field H_0. Fisher (1978) has shown that $\sigma = 1/\delta$ is a critical exponent of a one-component ϕ^3 theory when $d_{111} = 1$ and λ is pure imaginary. The critical exponent $\beta = 1$ for the Yang–Lee problem in all dimensions (de Alcantara Bonfim *et al* 1980).

In this letter we present a calculation in the context of renormalised perturbation theory of the beta function and critical exponents explicitly in three dimensions. This extends by a further two loops the work presented in Reeve *et al* (1982) in which at least encouraging, if not accurate, results were reported. Some minor numerical misprints and errors in that paper are corrected in the results that follow.

In order to remove the cut-off dependence and so to isolate the universal quantities (namely the critical exponents), the field theory defined in (1) by the Hamiltonian $\mathscr{H}$ is renormalised using the following usual conventions (Amit 1978):

$$Z_\phi \Gamma^{(2)}(q=0, m_0, \lambda) = \Gamma_{\mathrm{R}}^{(2)}(q=0, m, g) = m^2, \tag{7a}$$

$$Z_\phi^{3/2} \Gamma^{(3)}(q_i=0, m_0, \lambda) = \Gamma_{\mathrm{R}}^{(3)}(q_i=0, m, g) = g, \tag{7b}$$

$$\bar{Z}_{\phi^2} \Gamma^{(2,1)}(q_i=0, p=0, m_0, \lambda) = \Gamma_{\mathrm{R}}^{(2,1)}(q_i=0, p=0, m, g) = 1, \tag{7c}$$

$$Z_\phi \left.\frac{\partial \Gamma^{(2)}(q, m_0, \lambda)}{\partial q^2}\right|_{q^2=0} = \left.\frac{\partial \Gamma_{\mathrm{R}}^{(2)}(q, m, g)}{\partial q^2}\right|_{q^2=0} = 1, \tag{7d}$$

where $Z_\phi^{-1/2}$ is the wavefunction normalisation and m and g are respectively the renormalised mass and coupling constant. The function $\bar{Z}_{\phi^2}$ renders finite the vertex function with a mass operator insertion.

In terms of the dimensionless coupling constants $u = m^{-\varepsilon/2} g$ and $u_0 = m^{-\varepsilon/2}\lambda$ the Callan–Symanzik equation

$$[m\, \partial/\partial m + \beta(u)\, \partial/\partial u - \tfrac{1}{2} N \gamma_\phi(u)] \Gamma_{\mathrm{R}}^{(N)}(q_i, m, u) = (2 - \gamma_\phi(u)) m^2 \Gamma_{\mathrm{R}}^{N,1}(q_i=0, p=0, m, u) \tag{8}$$

expresses the independence of the unrenormalised vertex function from the particular normalisation point.

Assuming that the right-hand side of (8) is negligible (Amit 1976, Hubbard 1973) in the limit as $p_i/m \to \infty$, the exponent η is given by

$$\eta = \gamma_\phi(u^*) = \beta(u)\, \partial \ln Z_\phi / \partial u|_{u=u^*} \tag{9}$$

where u^* is such that

$$\beta(u^*) = \tfrac{1}{2}\varepsilon (\partial \ln u_0/\partial u)^{-1}|_{u=u^*} = 0. \tag{10}$$

In addition, the critical exponent ν is determined by

$$2-\eta-\nu^{-1}=\bar{\gamma}_{\phi^2}(u^*)=-\beta(u)\,\partial\ln\bar{Z}_{\phi^2}/\partial u|_{u=u^*}. \tag{11}$$

Also of interest is Wegner's correction-to-scaling exponent (Wegner 1972)

$$\omega=\partial\beta(u)/\partial u|_{u=u^*}. \tag{12}$$

Details of the method of calculation of the unrenormalised correlation functions are given in Reeve *et al* (1982). The evaluation of the Feynman integrals was made possible only by knowing the results for all one-loop diagrams as functions of the external momenta and internal masses (Nickel 1978).

The renormalisation group functions are then, for the percolation problem,

$$\beta(u)=-1.5u+1.3125u^3-1.407118u^5+2.98301u^7-7.482u^9+\ldots, \tag{13a}$$

$$\gamma_\phi(u)=-0.125u^2+0.061921u^4-0.016701u^6+0.3963u^8+\ldots, \tag{13b}$$

$$\bar{\gamma}_{\phi^2}(u)=0.75u^2-0.604166u^4+1.09228u^6-2.83547u^8+9.102u^{10}+\ldots, \tag{13c}$$

$$\gamma^{-1}(u)=1-0.375u^2+0.325521u^4-0.578096u^6+1.46708u^8-4.738u^{10}+\ldots, \tag{13d}$$

and for the Yang–Lee edge singularity problem

$$\beta(u)=-1.5u+0.5625u^3-0.274016u^5+0.247419u^7-0.2471u^9+\ldots, \tag{14a}$$

$$\gamma_\phi(u)=-0.125u^2+0.025655u^4-0.001677u^6+0.0133u^8+\ldots, \tag{14b}$$

$$\bar{\gamma}_{\phi^2}(u)=0.75u^2-0.3125u^4+0.249934u^6-0.267061u^8+0.3332u^{10}+\ldots, \tag{14c}$$

$$\gamma^{-1}(u)=1-0.375u^2+0.179688u^4-0.141008u^6+0.144963u^8-0.1801u^{10}+\ldots. \tag{14d}$$

The function

$$\gamma^{-1}(u)=1-\bar{\gamma}_{\phi^2}(u)/(2-\gamma_\phi(u)) \tag{15}$$

and the exponent γ is $\gamma(u^*)$.

We concentrate now on the analysis of the percolation problem. As has already been shown (Houghton *et al* 1978), $\beta(u)$ has an asymptotic expansion of the form

$$\beta(u)\sim C\sum_l(-a)^l\Gamma(l+b/2)u^{2l+1} \tag{16}$$

for large l. To sum the series $\beta(u)/u$ we first form the Borel transform by dividing each term of the series by $\Gamma(l+b/2)$. The Borel transform then undergoes a conformal transformation defined by

$$\lambda=[(1+au^2)^{1/2}-1]/[(1+au^2)^{1/2}+1]$$

which gives an expansion convergent in the entire u^2 cut plane. The integral representations of the $\Gamma(l+b/2)$ factors are then used to recover the original series. This follows identically the method of le Guillou and Zinn–Justin (1977, 1980), as used to analyse similar series for the ϕ^4, O(n) model (Baker *et al* 1976a, b).

Unfortunately we have been unable as yet to solve the instanton equation of Houghton *et al* (1978) needed to find an accurate value of a for our series. Instead we have estimated that $a=0.49\pm0.05$ by fitting the bare Green function to the form $C\,\Sigma_l\,u^{2l}a^l\Gamma(l+\frac{1}{2})$ and using a ratio test.

The results of the analysis to various orders are given in table 1, from which we estimate

$$\omega = 1.99 \pm 0.04, \qquad \eta = -0.131 \pm 0.001,$$
$$\bar{\gamma}_{\phi^2} = 0.909 \pm 0.010, \qquad \gamma = 1.74 \pm 0.015, \tag{17}$$

where the error estimates are not absolute, but simply reflect the variation in the fixed point u^* with a. The results (17) were all obtained independently and agree extremely well with the relation (15) and direct series estimates (Essam 1980). Unfortunately the Yang–Lee problem gave wildly inconsistent results when analysed by this method, as did both the Yang–Lee and percolation problems when analysed using the Padé–Borel method.

Table 1. Location of the fixed point and values of the critical exponents for successive orders of approximation for $a = 0.49$.

Function	u^*	ω	η	$\bar{\gamma}_{\phi^2}$	γ^{-1}
Order 4	2.870	1.21	−0.151	0.846	0.597
5	2.495	1.99	−0.131	0.894	0.581
6				0.909	0.575

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